A Bounded Rationality Approach to β, δ Preferences.

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Abstract

Borrowing from Cognitive Heierarchy Theory, I introduce bounded rationality into the β, δ model of present-biased preferences. I define a level-two agent—or "k-2–sophisticate"—as one who is aware that her future selves will have present-bias, but believes that they will be naive. The k-2–sophisticate does one round of strategic thinking about her future behavior instead of the unlimited number of rounds of the usual sophisticate. In the "doing it once" model of procrastination of O'Donoghue and Rabin (1999) the k-2–sophisticate typically procrastinates and preproperates less than the full sophisticate, and is protected from severe harm from both extreme preproperation and extreme procrastination, though she may suffer from excessive costly preemption due to pessimism about future preemption when costs are immediate.

1 Introduction

Behavioral economists have converged on the quasibyperbolic—or β, δ—model of Phelps and Pollack (1968) and Laibson (1997) to represent the psychological phenomenon of present-biased preferences, and explore issues of self control that may arise in the presence of such preferences. The predictions of the model frequently depend crucially on what assumptions are made about individuals' beliefs about their future preferences. O'Donoghue and Rabin (1999) worked out what has become the standard way of incorporating beliefs by introducing the $\widehat{\beta}$ parameter to capture naivete ($\widehat{\beta} = 1$), sophistication ($\widehat{\beta} = \beta$), and partial naivete ($\beta < \widehat{\beta} < 1$) with respect to future preferences.

There are times when this approach leads to results that seem counterintuitive or less than fully satisfactory. For example, one might hope that self knowledge would protect agents from severe harm, but in the "doing it once" setting of O'Donoghue and Rabin (1999) (hereafter O'D–R), complete sophistication can cause a mildly present-biased individual to experience severe welfare loss when benefits are immediate, where a naif with the same preferences would experience only mild harm. This is because a sophisticate is modeled as "unboundedly rational", in the sense that she is able to foresee an unlimited number of iterations of future behavior, and future foresight, right up to the terminal period. Put another way, in the O'D–R model the sophisticate's action in each period is determined through backward induction all the way from the terminal period, so that her pessimism about future self control can be compounded many times over. This observation leads to an obvious query: could more natural results be obtained by modeling foresight in a more natural way?

In this paper I borrow an idea from Cognitive Heierarchy Theory (CHT), which is to restrict the number of iterations of foresight that a sophisticated agent engages in. In CHT each player in a strategic game believes that the other players are less sophisticated, and therefore doing fewer rounds of strategic thinking, than themselves.[1] If we think of a

[1] See Camerer, Ho, and Chong (2004). I depart from their distributional assumption in modeling agents of level k as believing that all their future selves are level $k-1$.

discrete-time intertemporal model as a strategic game between a current self and a series of future selves then this kind of heierarchical approach can be applied quite naturally. In particular, in this paper I model time-consistent agents as level zero and naive agents—who believe their future selves will be time consistent—as level one. Then I introduce a new concept in intertemporal decision-making, the "k-2–sophisticate" who believes that all her future selves will be naive, or level one, and will thus be modeled as level two.[2] This approach allows for sophisticated beliefs about future preferences, while limiting the number of iterations of strategic thinking the sophisticated agent engages in. Extensive backward induction is no longer necessary, and the baleful pheneomenon of repeatedly compounded pessimism about future self control is mitigated.

I explore behavioral and welfare results for the k-2–sophisticate in the "doing it once" setting of O'D–R. I find that the k-2–sophisticate's behavior is qualitatively similar to O'D–R's full sophisticate, though the k-2–sophisticate procrastinates less than the full sophisticate when costs are immediate, and under natural restrictions on the evolution of delayed costs, preproperates less when rewards are immediate. In addition I find that, like the full sophisticate, the k-2–sophisticate with mild present bias is protected from disasterous procrastination when costs are immediate, but unlike the full sophisticate, is protected from disasterous preproperation when rewards are immediate. However, when costs are immediate she may engage in highly costly pre-emptive behavior due to excessive pessimism about future pre-emptive behavior, with the upper bound on harm, counter-intuitively, positively correlated with β. Section two reviews the O'D–R model and the behavioral and welfare results from that paper. Section three introduces k-2–sophistication and presents behavioral results. Section four presents welfare results for the k-2–sophisticate. Section five concludes.

[2]One could model levels above two, but they are less obviously natural in this setting than in game theory, and I do not explore them in this paper. It is worth noting, however, that in the limit as the level approaches infinity the CHT approach renders the O'Donoghue–Rabin full sophisticate. It is also worth noting that my CHT-based approach still allows for partial naivete as it includes the $\widehat{\beta}$ parameter to capture beliefs about future preferences.

2 Doing it Once: Setup and Results from O'D–R

Agents have periods $t = 1, 2, \ldots, T$ to do an action one time. Doing the action in period t renders reward v_t and cost c_t, one of which will be immediate and the other delayed. The vectors $\boldsymbol{v} = (v_1, v_2, \ldots, v_T)$ and $\boldsymbol{c} = (c_1, c_2, \ldots, c_T)$ fully define the setting. If rewards are immediate then $U^t(t)$, the agent's period-t instantaneous utility for doing it in period t, is $v_t - \beta c_t$ and if costs are immediate it is $\beta v_t - c_t$, while in either case $U^t(\tau)$, the period-t instantaneous utility of doing it in any period $\tau > t$, is $\beta(v_\tau - c_\tau)$, with $\beta \in [0, 1]$ capturing present bias.[3] Beliefs about future present bias are captured by $\widehat{\beta} \in [\beta, 1]$. Agents are of three types, $a \in \{TC, N, S\}$, for Time-Consistent ($\beta = 1$), Naive ($0 < \beta < 1$ and $\widehat{\beta} = 1$), and Sophisticated ($0 < \beta < 1$ and $\widehat{\beta} = \beta$). An agent's strategy, $\boldsymbol{s} = (s_1, s_2, \ldots, s_T)$, with $s_t \in \{Y, N\}$, describes whether she will do it in each period conditional on not having done it already.

Solution concepts for the three types are based on the principle that each period's choice must be optimal with respect to what the agent believes she will do in the future. O'D–R define "perception perfect" strategies for the three types. Actual behavior for an agent of type a in any given setting is to do it in the first period for which $s_t^a = Y$. That period is refered to as τ_a.

Definition 1 (O'D–R 2) *A perception perfect strategy for TCs is a strategy* $\boldsymbol{s}^{tc} \equiv \left(s_1^{tc}, s_2^{tc}, \ldots s_T^{tc}\right)$ *that satisfies for all* $t < T$ $s_t^{tc} = Y$ *if and only if* $U^t(t) \geq U^t(\tau)$ *for all* $\tau > T$.

Definition 2 (O'D–R 3) *A perception perfect strategy for naifs is a strategy* $\boldsymbol{s}^n \equiv (s_1^n, s_2^n, \ldots s_T^n)$ *that satisfies for all* $t < T$ $s_t^n = Y$ *if and only if* $U^t(t) \geq U^t(\tau)$ *for all* $\tau > T$.

Definition 3 (O'D–R 4) *A perception perfect strategy for sophisticates is a strategy* $\boldsymbol{s}^s \equiv (s_1^s, s_2^s, \ldots s_T^s)$ *that satisfies for all* $t < T$ $s_t^s = Y$ *if and only if* $U^t(t) \geq U^t(\tau')$, *where* $\tau' \equiv \min_{\tau > t} \{\tau \mid s_\tau^s = Y\}$.

A time-consistent agent does it in the period with the highest net benefit. A naif does

[3] For simplicity O'D–R let $\delta = 1$.

it in the first period which his taste for immediate gratification tells him is better than all future periods. A sophisticate does it in the first period that her taste for immediate gratification tells her is better than all future periods in which her future self would do it, given what she foresees about what her future selves will foresee about what subsequently future selves will foresee about... You get the point. The solution concept for sophisticates requires $T - t$ iterations of "strategic" thinking in every period.

The examples in O'D–R elucidate these solution concepts. A cinema shows one film each Saturday for four weeks with ascending values of 3, 5, 8, and 13. In the first example, of immediate costs, agents must miss a film to complete a report on one of the four Saturdays, rendering delayed reward of $\bar{\nu}$. In the second example, of immediate rewards, agents have a coupon good for one film and cannot see more than one, and delayed cost is normalized to zero. In both examples we explore the behavior of TCs, and of naifs and sophisticates with $\beta = \frac{1}{2}$.

Example 1 (O'D–R 1) *Immediate costs:* $\boldsymbol{v} = (\bar{\nu}, \bar{\nu}, \bar{\nu}, \bar{\nu})$ $\boldsymbol{c} = (3, 5, 8, 13)$

$\boldsymbol{s}^{tc} = (Y, Y, Y, Y), \ \tau_{tc} = 1$

$\boldsymbol{s}^{n} = (N, N, N, Y), \ \tau_{n} = 4$

$\boldsymbol{s}^{s} = (N, Y, N, Y), \ \tau_{s} = 2.$

The TC does the report promptly, the naif procrastinates disasterously, the sophisticate procrastinates less.

Example 2 (O'D–R 2) *Immediate rewards:* $\boldsymbol{v} = (3, 5, 8, 13)$ $\boldsymbol{c} = (0, 0, 0, 0)$

$\boldsymbol{s}^{tc} = (N, N, N, Y), \ \tau_{tc} = 4$

$\boldsymbol{s}^{n} = (N, N, Y, Y), \ \tau_{n} = 3$

$\boldsymbol{s}^{s} = (Y, Y, Y, Y), \ \tau_{s} = 1.$

The TC exercises full restraint, the naif preproperates a bit, the sophisticate preproperates disasterously.

Why does the sophisticate fare so badly in example 2? To decide whether to see the first movie she has to figure out which future movies she will go to if she skips the first.

This involves putting herself into the shoes of her period two self, but to figure out what she'll do next week she has to put herself into the shoes of her period three self. In each case she foresees a future of one-period-at-a-time preproperation and in despair mopes off to see the worst film.

O'D–R next demonstrate that this pattern of behavior is quite general.[4]

Proposition 1 (O'D–R 1) *(1) If costs are immediate, then* $\tau_n \geq \tau_{tc}$. *(2) If rewards are immediate, then* $\tau_n \leq \tau_{tc}$.

The naif always does the wrong thing relative to the TC, which O'D–R call the present-bias effect.

Proposition 2 (O'D–R 2) *For all cases,* $\tau_s \leq \tau_n$.

The sophisticate foresees the trouble her present bias will cause her in the future and either procrastinates less or preproperates more—which O'D–R call the sophistication effect—in both cases because she realizes that some prefered future period is not a real option.

Furthermore, O'D–R show that the pattern of potential harm implied by the examples is also quite general. Restricting attention to settings in which there is an upper bound, $\bar{X}$, to the reward and/or cost of any given period they work out the worst-case scenarios for naive and sophisticated agents with arbitrarily mild present bias. Their welfare comparisons are based on a long-term view of utility, which is mathematically the same as utility for a time-consistent agent. Notationally, the long-term utility of period t is $U^0(t) \equiv v_t - c_t$

Proposition 3 (O'D–R 3) *Suppose costs are immediate and consider all* $\boldsymbol{v}$ *and* $\boldsymbol{c}$ *such that* $v_t \leq \bar{X}$ *and* $c_t \leq \bar{X}$ *for all* t*:*

(1) $\lim_{\beta \to 1}(\sup_{(\boldsymbol{v},\boldsymbol{c})}[U^0(\tau_{tc}) - U^0(\tau_s)]) = 0,$ *and*

(2) For any $\beta < 1,\ \sup_{(\boldsymbol{v},\boldsymbol{c})}[U^0(\tau_{tc}) - U^0(\tau_n)] = 2\bar{X}$.

In certain settings even a minutely present-biased naif may put off the task repeatedly, always thinking he will do it in the next most prefered period, incurring only a small

[4] Proofs of O'D–R's results can be found in the appendix to their paper.

welfare cost each time, but eventually losing all. A sophisticate with the same preferences will always accurately foresee her entire strategy. If she doesn't do it in τ_{tc} it can only be because her present bias convinces her τ_{tc} is less desirable than some other period when she actually does do it, and because her present bias is tiny, the difference between that period and τ_{tc} must also be tiny.

Proposition 4 (O'D–R 4) *Suppose rewards are immediate and consider all $\boldsymbol{v}$ and $\boldsymbol{c}$ such that $v_t \leq \bar{X}$ and $c_t \leq \bar{X}$ for all t:*

(1) $\lim_{\beta \to 1}(\sup_{(\boldsymbol{v},\boldsymbol{c})}[U^0(\tau_{tc}) - U^0(\tau_n)]) = 0,$ *and*

(2) For any $\beta < 1,\ \sup_{(\boldsymbol{v},\boldsymbol{c})}[U^0(\tau_{tc}) - U^0(\tau_n)] = 2\bar{X}.$

A naif always thinks he will do it in τ_{tc} and thus compares each period to that most prefered period. Thus, if he is only minutely present biased then he will do it in a period that is only minutely less prefered than τ_{tc}. In certain settings a sophisticate with the same preferences will foresee an unwinding backward sequence of future selves foreseeing that their future selves will preproperate, and therefore do it in the worst period because her present bias makes her think it is just marginally better than her next-best realistic alternative.

In the same way that some results in game theory which involve agents doing many rounds of strategic thinking are unsatisfactory, this catastrophic outcome for a minutely present biased sophisticate, relying as it does upon many rounds of pessimistic foresight, leaves something to be desired. It seems intuitively reasonable that a drastically present-biased sophisticate could second-guess herself and do the task in a drastically sub-optimal period. But for a minutely present-biased to do so seems counter-intuitive. And it is the assumption of unbounded rationality that is driving the odd result.

3 K-2–sophistication: Definition and Behavior

The crucial step in applying Cognitive Heierarchy Theory to a novel setting is to define level-zero behavior, as all other levels are defined in terms of this single building block. The natural starting place in the β, δ is time consistency, which involves no consideration of future selves preferences. Careful inspection of definition 2 reveals that a naif thinks that all his future selves will be time consistent, or level zero, so in the framework of CHT a naif is level one. Taking things to the next level, a level two agent thinks all of her future selves will be level one, or naive. This allows for foresight with respect to present-biased preferences, while introducing bounded rationality with respect to the number of iterations of foresight a sophisticate engages in. For this reason I refer to a level two agent as a "k-2–sophisticate".[5] Thus, a k-2–sophisticate does it in any period that appears better than the next period in which a naif would do it in. And since a naif's behavior can always be determined prospectively, so can that of a k-2–sophisticate. To formalize these concepts:

Definition 4 *A perception perfect strategy for k-2–sophisticates is a strategy* $\boldsymbol{s}^k \equiv (s_1^k, s_2^k, \ldots s_T^k)$ *that satisfies for all* $t < T \quad s_t^k = Y$ *if and only if* $U^t(t) \geq U^t(\tau')$*, where* $\tau' \equiv \min_{\tau > t}\{\tau \mid s_\tau^n = Y\}$.

If she has not done it already, a k-2–sophisticate will do it in period t if and only if the utility of doing so is greater than the perceived (beta-discounted) utility of doing it in the next period in which a naif would do it.

The goal of the exercise is to preserve the qualitative behavioral results of sophistication while improving the welfare results. I begin by exploring behavioral results.

Proposition 5 *For all cases,* $\tau^k \leq \tau^n$.

The k-sophisticate always does it as soon or sooner than the naif. Thus the sophistication effect of O'D–R is preserved under k-2–sophistication.

The behavioral comparison between the k-2–sophisticate and the full sophisticate is slightly more complicated. The following proposition addresses results for a limited but interesting set of cases.

[5] The "k" comes from the terminology of CHT, in which k refers to the level of an agent.

Proposition 6 *(1) If rewards are immediate and $c_t \geq c_{t+1}$ for all t, then $\tau^s \leq \tau^k$. (2) If costs are immediate, then $\tau^k \leq \tau^s$*

What proposition 2b says is that when delayed costs are constant or decreasing, and for any sequence of delayed benefits, the k-2–sophisticate does less of the bad thing than the full sophisticate. She preproperates less because she does only one round of strategic thinking and thus avoids the tragedy of endless second guessing that causes the full sophisticate to abandon any hope of exerting self-control.[6] She procrastinates less because, once again doing only one round of strategic thinking, she compares each period to a worst-case scenario that the full sophisticate knows she won't actually have to face.

We can see proposition 6 in action by looking at what a k-2–sophisticate would do in the cinema examples of O'D–R.

Example 3 *A k-2–sophisticate goes to the cinema.*
(1) In the immediate costs setting of example 1 we have $\boldsymbol{s}^k = (Y, Y, Y, Y),\ \tau_k = 1$.
(2) In the immediate rewards setting of example 2 we have $\boldsymbol{s}^k = (N, Y, Y, Y),\ \tau_k = 2$.

In keeping with 6, when costs are immediate the k-2–sophisticate procrastinates less than the full sophisticate because she is more pessimistic about her future self-control. In period one she says, "I know myself. I'll put this off until the last moment and miss the best film. I need to get it out of the way now or all hope will be lost." It is true that she knows herself, in the sense that she knows she has a persistent problem with self-control, but it is also true that she applies that self-knowledge to the consideration of her future behavior in a limited way. In this case it works in her favor. When rewards are immediate she sees the film in period two because she foresees herself preproperating in period three. But in period one she does not foresee her period-two preproperation because she is only thinking of what a naif would do, which is to see the film in the third period. She knows herself, but not fully. In this case, once again, bounded rationality works in her favor.

[6] I consider the limited set of cases in which a k-2–sophisticate preproperates more than a full sophisticate in an appendix.

4 Welfare

The O'D–R cinema examples are ideal for the k-2–sophisticate, giving her a better welfare outcome than the full sophisticate whether costs or rewards are immediate. As O'D–R point out in their paper, fully general welfare comparisons are prohibitively complicated. However, a couple of examples will show how things can backfire on the k-2–sophisticate, relative to the full sophisticate. First, imagine adding to example 1 an additional week, at the beginning, when the cinema is playing quite a good film, worth 6.

Example 4 *Immediate costs:* $\boldsymbol{v} = (\bar{\nu}, \bar{\nu}, \bar{\nu}, \bar{\nu}, \bar{\nu})$ $\boldsymbol{c} = (6, 3, 5, 8, 13)$

$\boldsymbol{s}^{tc} = (N, Y, Y, Y, Y),\ \tau_{tc} = 2$

$\boldsymbol{s}^{n} = (N, N, N, N, Y),\ \tau_{n} = 5$

$\boldsymbol{s}^{s} = (N, N, Y, N, Y),\ \tau_{s} = 3.$

$\boldsymbol{s}^{k} = (Y, Y, Y, Y, Y),\ \tau_{k} = 1$

The addition of the quite good film doesn't change the behavior of the time consistent agent, the naif, or the full sophisiticate. But the k-2–sophisticate, in the first period, because she does not think through what she will do in periods four or three, thinks her only chance to get the better of her impulsive future self is to get the report out of the way immediately. One way to think of this is that, though she is less pessimistic about her future self control problems than the full sophisticate, she is more pessimistic about her future preemtive behavior. As we will see, this kind of excessive preemption of procrastination is the only way that a k-2–sophisticate with mild present bias can get hurt badly.

When rewards are immediate there are also cases where the k-2–sophisticate fares worse than the full sophisticate. Consider the film-coupon setup of example 2 and imagine that a large conference has been planned at a nearby hotel for the third week. The cinema has decided to maximize the take from conference goers by reducing the value of the coupons they give out to locals, requiring them to pay a portion of the ticket price that week worth 4. In addition, to make the example work, imagine that the first film is worth 2.25 and the last, 11.

Example 5 *Immediate rewards:* $\boldsymbol{v} = (2.25, 5, 8, 11)$ $\boldsymbol{c} = (0, 0, 4, 0)$

$\boldsymbol{s}^{tc} = (N, N, N, Y),\ \tau_{tc} = 4$

$\boldsymbol{s}^{n} = (N, N, Y, Y),\ \tau_{n} = 3$

$\boldsymbol{s}^{s} = (N, Y, Y, Y),\ \tau_{s} = 2$

$\boldsymbol{s}^{k} = (Y, Y, Y, Y),\ \tau_{k} = 1.$

Both the naif and the sophisticate go to the film in week three, which means that in week two both the k-2–sophisticate and the full sophisticate go to the film. However, in week one the sophisticate foresees that she'll get the better deal of week two while the k-2–sophisticate still has her eyes on week three because she hasn't worked out that the added cost that week will make her want to go in week two. Again, what hurts the k-2–sophisticate in this case is her excessive pessimism about future self control. She consistently fails to predict the positive steps her future selves will be willing to take to manage her self control problem. However, as we will see, in the case of immediate rewards this kind of mistake cannot cause greivious harm to a k-2–sophisticate with only mild present bias.

Following O'D–R I next consider worst-case welfare scenarios when present bias is mild. The essence of their welfare results is the number of rounds of self-destructive decision making or strategic thinking that agents engage in. When costs are immediate the naif is capable of procrastinating over and over again, hurting himself each time by an amount that is bounded by a diminishing function of β, but potentially accumulating a large welfare loss over many periods of iterative decision making. The full sophisticate avoids all of this iteration by accurately foreseeing all of the periods she might do it and choosing the one she likes best. Only one round of self-destructive decision making, the cost of which is bounded by that same diminishing function of β, so that serious harm can only come to an agent with a serious self-control problem. Meanwhile, when benefits are immediate the naif does it in the first period that looks better than τ_{TC}, one round of decision making and again, the amount of his welfare loss from that single round of decision making is bounded by a diminishing function of β, so he can't get that badly hurt unless he has an overwhelming self-control problem. The full sophisticate, however, is capable of engaging in an unlimited number of rounds of pessimistic backward induction about her future behavior, concluding, with each round of strategic thinking, that her preproperation will cause her to do it earlier and earlier, and leading, potentially, to extreme preproperation and large welfare loss.

By contrast, the mildly present-biased k-2–sophisticate is protected from the naif's many rounds of procrastination by her foresight, and from the full sophisticates many iterations of pessimistic foresight by her bounded rationality. The only serious harm she can come to is excessive preemption of procrastination. First we consider the procrastination result.

Proposition 7 *Suppose costs are immediate and consider all* $\boldsymbol{v}$ *and* $\boldsymbol{c}$ *such that* $v_t \leq \bar{X}$ *and* $c_t \leq \bar{X}$ *for all* t*:*

(1) $[\tau_k \geq \tau_{tc}] : \lim_{\beta \to 1}(\sup_{(\boldsymbol{v},\boldsymbol{c} \mid \tau_k \geq \tau_{tc})}[U^0(\tau_{tc}) - U^0(\tau_k)]) = 0$

(2) $[\tau_k < \tau_{tc}]$ *: For any* $\beta < 1$, $\sup_{(\boldsymbol{v},\boldsymbol{c} \mid \tau_k < \tau_{tc})}[U^0(\tau_{tc}) - U^0(\tau_k)] = (1+\beta)\bar{X}$

Whenever a mildly present-biased k-2–sophisticate hasn't done it before τ_{tc}, if she doesn't do it in τ_{tc} it must be because τ_n is not that much worse than τ_{tc} and since τ_k has to be weakly better than τ_n the welfare loss is bounded and vanishes as β approaches one.[7]

However, in cases where a time-consistent agent doesn't do it in the first period, difficulty may arise for the k-2–sophisticate, even when present bias is mild. The k-2–sophisticate looks at the horrendous outcome that she believes lies in wait for her and, believing that she won't do it at, or after, τ_{tc}, she does it in the first period that feels better in the short-term than her discounted assesment of τ_n, which may be a much less desirable period than τ_{tc}. However, she is protected by her present bias. If she has very mild present bias then she is realistic about how painful τ_n is going to be, and will be willing to do it in an almost as painful early period. If, instead, she has substantial present bias then she erroneously believes that τ_n will not be so bad, and thus passes over very painful early periods and only does it in an early period if the short term cost is relatively low. Thus, ironically, as β approaches one the k-2–sophisticate may lose everything.

[7]It may be worth noting that in the case of constant or diminishing (check this) delayed rewards we get $\tau_k = \tau_n$ because in this case $\beta v_t - c_t \leq \beta v_{\tau_{tc}} - c_{\tau_{tc}}$, $\forall t$, and in particular, for $\tau_{tc} < t < \tau_n$, $\beta v_t - c_t \leq \beta v_{\tau_{tc}} - c_{\tau_{tc}} < \beta U^0(\tau_n)$.

Next I consider the case of immediate rewards.

Proposition 8 *Suppose rewards are immediate and consider all* $\boldsymbol{v}$ *and* $\boldsymbol{c}$ *such that* $v_t \leq \bar{X}$ *and* $c_t \leq \bar{X}$ *for all* t*:*

$\lim_{\beta \to 1}(\sup_{(\boldsymbol{v},\boldsymbol{c})}[U^0(\tau_{tc}) - U^0(\tau_k)]) = 0$

When rewards are immediate the k-2–sophisticate with mild present bias cannot be severly harmed by extreme preproperation. The naif does one round of preproperation, and foreseeing this, the k-sophisticate does one more round of preproperation. In each of these rounds the welfare loss is limited as a function of β. The thing that can lead the full sophisticate to ruin is that she is capable of foreseeing an unlimited number of iterations of preproperation and the accumulation of small welfare losses can become severe.

5 Conclusion

Introducing bounded rationality into a model of present-biased preferences by borrowing from Cognitive Heierarchy Theory appears to render more natural results for procrastination and preproperation in a setting where an agent must do a task with either immediate costs or immediate rewards one time in a fixed number of periods. A "boundedly rational" k-2–sophisticate typically preproperates less, and always procrastinates less, than an "unboundedly rational" full sophisticate, while still exhibiting the sophistication effect of always doing the task before a naif. When present-bias is mild, like the naif, the k-2–sophisticate is protected from extreme preproperation, and like the full sophisticate, is protected from extreme procrastination, except in cases of excessive preemptive behavior.

This is a very preliminary exploration of the role of bounded rationality in models of present bias. An important step would be to review existing results for full sophistication in various models and see whether k-2–sophistication preserves and/or improves those results. In particular, it would be very helpful to know whether limiting the number of rounds of prospective thinking sophisticated agents engage in, and thus largely obviating backward induction, could lead to unique solutions in infinite-horizon settings where full sophistication often leads to multiple solutions. It may also be worth exploring levels of cognitive heierarchy higher than two.

One of the interesting features of the CHT approach I have developed in this paper is that it separates agents' beliefs about their future preferences from their beliefs about their future beliefs. In the O'D–R model the $\widehat{\beta}$ parameter does double duty by simultaneously capturing beliefs about future preferences and beliefs about future beliefs. If a decision maker has a preference parameter β, the model tells us not only that she believes her future selves will have a preference parameter of $\widehat{\beta}$ but also that she believes her future selves will have the same belief about their respective future-selves' preferences. By contrast, a k-2–sophisticate believes that her future selves will have preference parameter $\beta = \widehat{beta}$ but belief parameter $\widehat{beta} = 1$. It could be useful to explore other approaches to separting beliefs about preferences from beliefs about beliefs.

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A Appendices

A.1 When does k preproperate more than s?

$\tau_k < \tau_s$ requires that there be some period when k does it and s doesn't, which means $U^{\tau_k}(\tau'_k) \leq U^{\tau_k}(\tau_k) < U^{\tau_k}(\tau'_s)$. Rewriting the ends of this double inequality gives us $v_{\tau'_s} - c_{\tau'_s} > v_{\tau'_k} - c_{\tau'_k}$, which we can rearrange to get $v_{\tau'_s} - v_{\tau'_k} > c_{\tau'_s} - c_{\tau'_k}$ Next, notice that $\tau_k < \tau_s$ requires that in τ_k we have τ'_s strictly before τ'_k, which means that a naif would not do it in τ'_s. Now, the only reason this can be true is if there is some period, say t', after τ'_k in which the naif, in τ'_s thinks she will do it.[8] This requires $U^{\tau'_s}(t') > U^{\tau'_s}(\tau'_s)$ which, by the definition of τ'_k requires $U^{\tau'_k}(\tau'_k) > U^{\tau'_s}(\tau'_s)$ which gives us $v_{\tau'_s} - \beta c_{\tau'_s} > v_{\tau'_k} - \beta c_{\tau'_k}$. Rearranging this and combining with the earlier result we get the full condition:

$$c_{\tau'_s} - c_{\tau'_k} < v_{\tau'_s} - v_{\tau'_k} < \beta(c_{\tau'_s} - c_{\tau'_k})$$

Notice that this double inequality can only hold when costs are increasing between τ'_s and τ'_k, and in particular, increasing more than rewards, but not too much more.

A.2 Proofs

Proof of Proposition 5.

Recall that the naif does it in period t if and only if $U^t(t) \geq U^t(\tau)$ for all $\tau > t$, while the the k-2–sophisticate does it in period t if and only if $U^t(t) \geq U^t(\tau')$. Since $\{\tau'\} \subseteq \{\tau \mid \tau > t\}$ and the maximum of a subset is weakly less than the maximum of the superset, the k-2–sophisticate does it whenever the naif does, and in particular may do it when the naif doesn't, i.e. sooner.

Proof of Proposition 6.

Let $t < T$ be an arbitrary, non-terminal period. Relative to t we refer to the τ' in definition 3 as τ'_s and the τ' in definition 4 as τ'_k.

(1) By proposition 2 $\tau'_s \leq \tau'_k$. The proof consists of showing that $U^t(\tau'_s) \geq U^t(\tau'_k)$ so that if k does it in period t, s does too, and may do it when k does not. Now $v_{\tau'_k-1} - \beta c_{\tau'_k-1} < v_{\tau'_k} - \beta c_{\tau'_k}$ because if not the definition of τ'_k is contradicted. To see this, notice that by the definition of τ'_k we have $v_{\tau'_k} - \beta c_{\tau'_k} \geq \max_{\tau > \tau'_k}\{\beta(v_\tau - c_\tau)\}$, and since $v_{\tau'_k} - \beta c_{\tau'_k} > \beta(v_{\tau'_k} - c_{\tau'_k})$, if $v_{\tau'_k-1} - \beta c_{\tau'_k-1} \geq v_{\tau'_k} - \beta c_{\tau'_k}$ then $s^n_{\tau'_k-1} = Y$ which contradicts the definition of τ'_k. By iteration, $v_\tau - \beta c_\tau < v_{\tau'_k} - \beta c_{\tau'_k}$, for all $t < \tau < \tau'_k$ and since for all t $c_t \geq c_{t+1}$ we get $v_\tau - c_\tau < v_{\tau'_k} - c_{\tau'_k}$ for all $t < \tau < \tau'_k$ which means $v_{\tau'_s} - c_{\tau'_s} \leq v_{\tau'_k} - c_{\tau'_k}$. Thus $U^t(t) \geq U^t(\tau'_k) \implies U^t(t) \geq U^t(\tau'_s)$, which means s does it whenever k does it.

[8] Need to check this assertion. Basically it has to be the case that if n doesn't do it at tauprime s it must be because there's some period she thinks will be better, so I just need to show that that period cannot come before tauprime k without violating the definition of tauprime k.

(2) The proof consists of showing that $U^t(\tau'_k) \geq U^t(\tau'_s)$ so that if s does it in period t, k does too, and may do it when s does not. By proposition 2 we have $\tau'_s \leq \tau'_k$ and because s does it whenever n does it, $s^s_{\tau'_k} = Y$. By the definition of τ'_s we have $\beta v_{\tau'_s} - c_{\tau'_s} \geq \beta(v_{\tau'_k} - c_{\tau'_k})$, and since $\beta c_{\tau'_s} < c_{\tau'_s}$ we have $v_{\tau'_s} - c_{\tau'_s} \geq v_{\tau'_k} - c_{\tau'_k}$. Thus $U^t(t) \geq U^t(\tau'_s) \implies U^t(t) \geq U^t(\tau'_k)$, which means k does it whenever s does it.

Proof of proposition 7

(1) If $\tau_k = \tau_{tc}$ then $U^0(\tau_{tc}) - U^0(\tau_k) = 0$. If $\tau_k > \tau_{tc}$ we know from proposition 5 that $\tau_k \geq \tau_n$ and by the definition of τ_k we have $\beta v_{\tau_k} - c_{\tau_k} \geq \beta v_{\tau_n} - \beta c_{\tau_n}$, and since $\beta v_{\tau_k} - \beta c_{\tau_k} \geq \beta v_{\tau_k} - c_{\tau_k}$ we get $U^0(\tau_k) \geq U^0(\tau_n)$. Now $\tau_k > \tau_{tc} \implies s^k_{\tau_{tc}} = N \implies \beta v_{\tau tc} - c_{\tau_{tc}} < U^0(\tau_n) \leq U^0(\tau_k)$. Rearranging we get $\beta U^0(\tau_{tc}) - (1-\beta)c_{\tau_{tc}} < \beta U^0(\tau_k)$ and rearranging again we get $0 \leq U^0(\tau_{tc}) - U^0(\tau_k) < \frac{1-\beta}{\beta} c_{\tau_{tc}} \leq \frac{1-\beta}{\beta}\bar{X}$, where the first inequality arises from the definition of τ_{tc} as the period with the highest ex-ante utility. Hence $0 \leq \sup_{(\boldsymbol{v},\boldsymbol{c} \mid \tau_k \geq \tau_{tc})}[U^0(\tau_{tc}) - U^0(\tau_k)] < \frac{1-\beta}{\beta}\bar{X}$ and the result follows from the squeeze theorem.

(2) $U^0(\tau_{tc}) - U^0(\tau_k) = [U^0(\tau_{tc}) - U^0(\tau_n)] - [U^0(\tau_k) - U^0(\tau_n)]$ By proposition 3 we know that $\sup_{(\boldsymbol{v},\boldsymbol{c})}[U^0(\tau_{tc}) - U^0(\tau_n)] = 2\bar{X}$ and from the proof of that proposition in O'D–R we know that the welfare loss converges to this supremum when $(v_{\tau_{tc}}, c_{\tau_{tc}}, v_{\tau_n}, c_{\tau_n}) = (\bar{X}, \varepsilon, 0, \bar{X})$, where $\varepsilon \in (0, \bar{X})$ is some arbitrarily small positive number. Now let us add a period before $v_{\tau_{tc}}$ and call this period 1, and let $v_1 = 0$, and $c_1 = \beta\bar{X}$ so that $s^k_1 = Y$, $\tau_k = 1$, and $U^0(\tau_k) = -\beta\bar{X}$. Thus $U^0(\tau_k) - U^0(\tau_n) = -(\beta\bar{X}) - (-\bar{X}) = (1-\beta)\bar{X}$. As this is the smallest value of $U^0(\tau_k) - U^0(\tau_n)$ for which $\tau_k < \tau_{tc}$ we have $\sup_{(\boldsymbol{v},\boldsymbol{c} \mid \tau_k < \tau_{tc})} -[U^0(\tau_k) - U^0(\tau_n)] = (1-\beta)\bar{X}$ and since this supremum and the one above are both approached by the same $(\boldsymbol{v}, \boldsymbol{c})$ vector we get $\sup_{(\boldsymbol{v},\boldsymbol{c} \mid \tau_k < \tau_{tc})}[U^0(\tau_{tc}) - U^0(\tau_k)] = 2\bar{X} - (1-\beta)\bar{X} = (1+\beta)\bar{X}$.

Proof of Proposition 8.

$U^0(\tau_{tc}) - U^0(\tau_k) = [U^0(\tau_{tc}) - U^0(\tau_n)] + [U^0(\tau_n) - U^0(\tau_k)]$ By the definition of τ_n we know that $U^0(\tau_{tc}) - U^0(\tau_n) \leq \frac{1-\beta}{\beta} v_{\tau_n} \leq \frac{1-\beta}{\beta}\bar{X}$. (This is derived in the proof of prosposition 4.1 in O'D–R.) If $\tau_k = \tau_n$ then $U^0(\tau_n) - U^0(\tau_k) = 0$. Otherwise, by the definition of τ_k, we have $v_{\tau_k} - \beta c_{\tau_k} > \beta U^0(\tau_n)$ which by rearranging gets us $U^0(\tau_n) - U^0(\tau_k) < \frac{1-\beta}{\beta} v_{\tau_k} \leq \frac{1-\beta}{\beta}\bar{X}$. Thus we get that $0 \leq U^0(\tau_{tc}) - U^0(\tau_k) \leq 2\frac{1-\beta}{\beta}\bar{X}$ which implies $0 \leq \sup_{(\boldsymbol{v},\boldsymbol{c})}[U^0(\tau_{tc}) - U^0(\tau_k)] \leq 2\frac{1-\beta}{\beta}\bar{X}$, and the result follows from the squeeze theorem.